5.3 Solution Curves / Phase Portraits

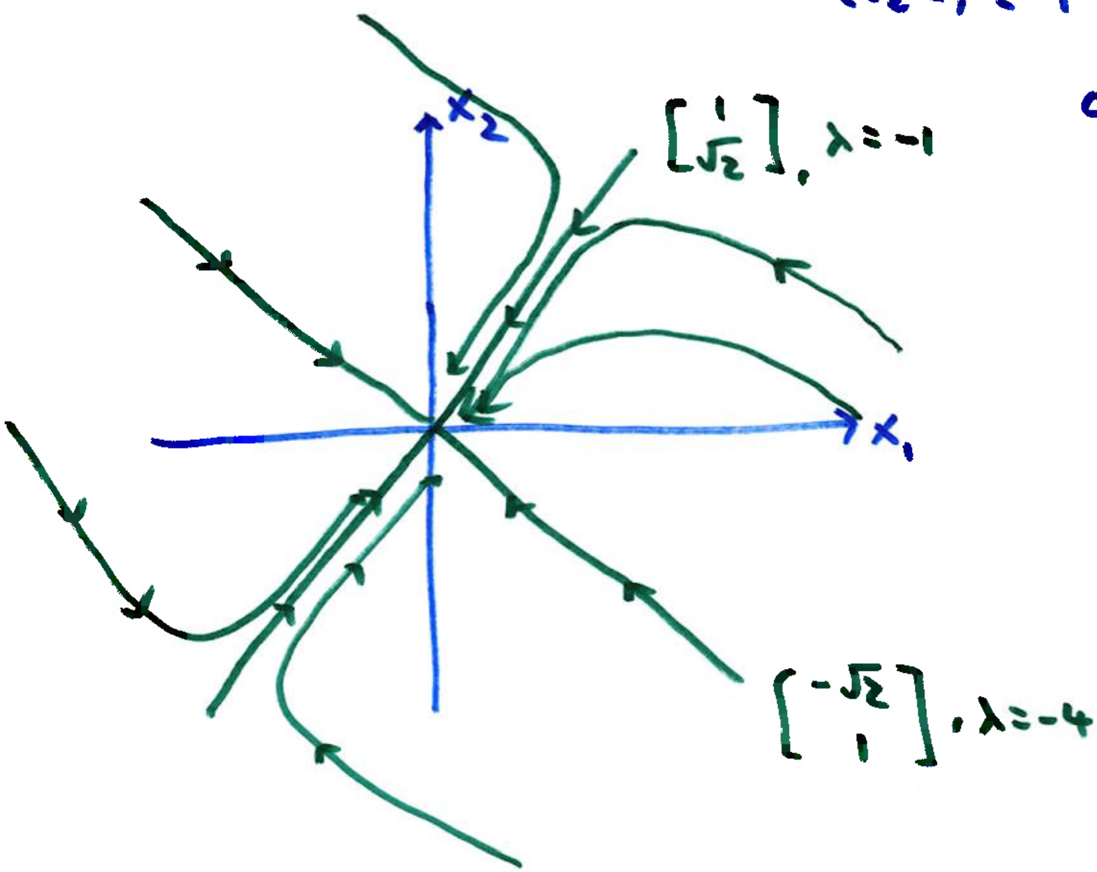
summary of $\vec{x}\,' = A\vec{x}$

λ's real and distinct

$$\vec{x}\,' = \begin{bmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{bmatrix}\vec{x}$$

$\lambda = -1, -4$

$\vec{v} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}, \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$ ← straight line solutions



origin: $t = \infty$ ($e^{\lambda t} \to 0$ as $t \to \infty$ if $\lambda < 0$)

at $t = \infty$, $e^{-t} > e^{-4t}$

so near origin, solution curves follow the eigenvector of $\lambda = -1$

origin is an equilibrium

also an <u>improper nodal sink</u>

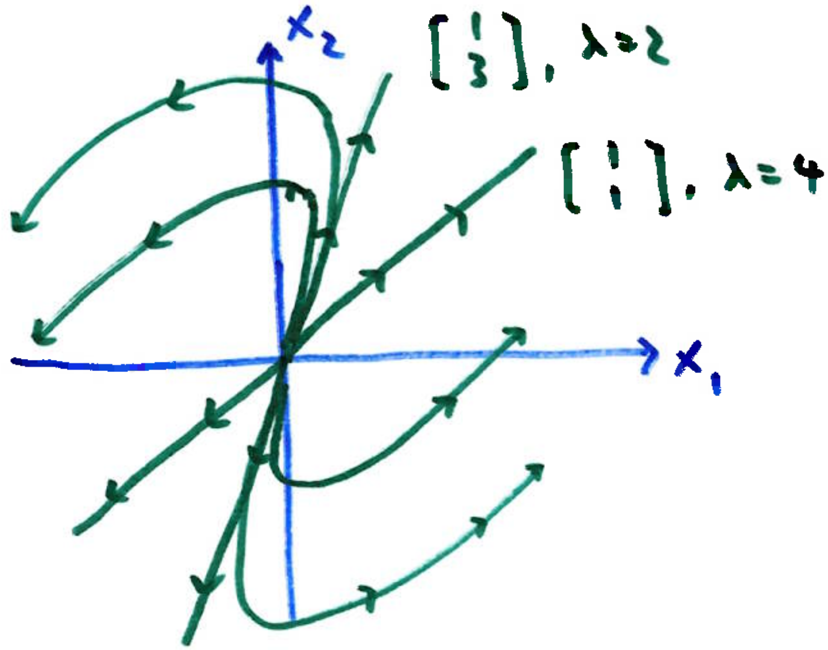
solutions go into origin

"point"

along an asymptote into/out of origin

$\vec{x}' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \vec{x}$ $\lambda = 2,\ 4$

$\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



origin: $t = -\infty$

$e^{2t} > e^{4t}$

$\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ leave along this near origin

origin is an

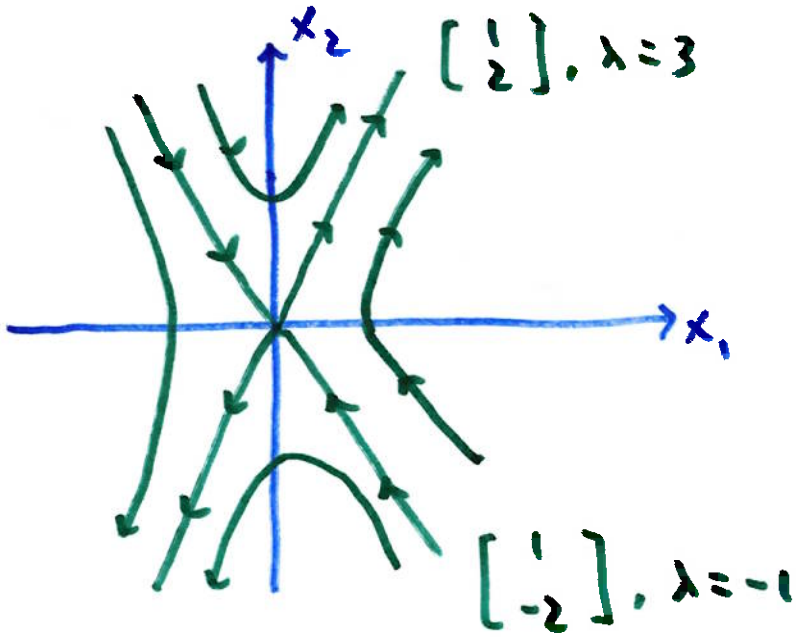
<u>improper nodal source</u>

λ's real and opposite in signs

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \vec{x}$$

$\lambda = 3, \; -1$

$\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$



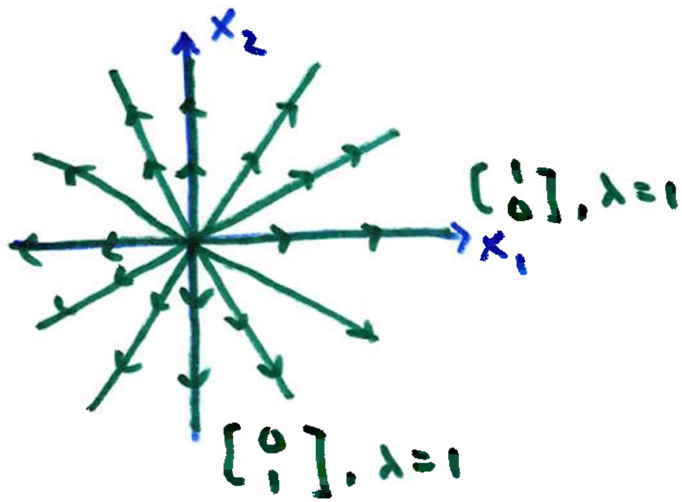
origin is a saddle point

λ's are repeated

$$\vec{x}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x} \qquad \lambda = 1, 1 \qquad \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(matrix is complete)

not defective



eigenvalues are equal, so neither eigenvector is more important → solutions are linear combos of both

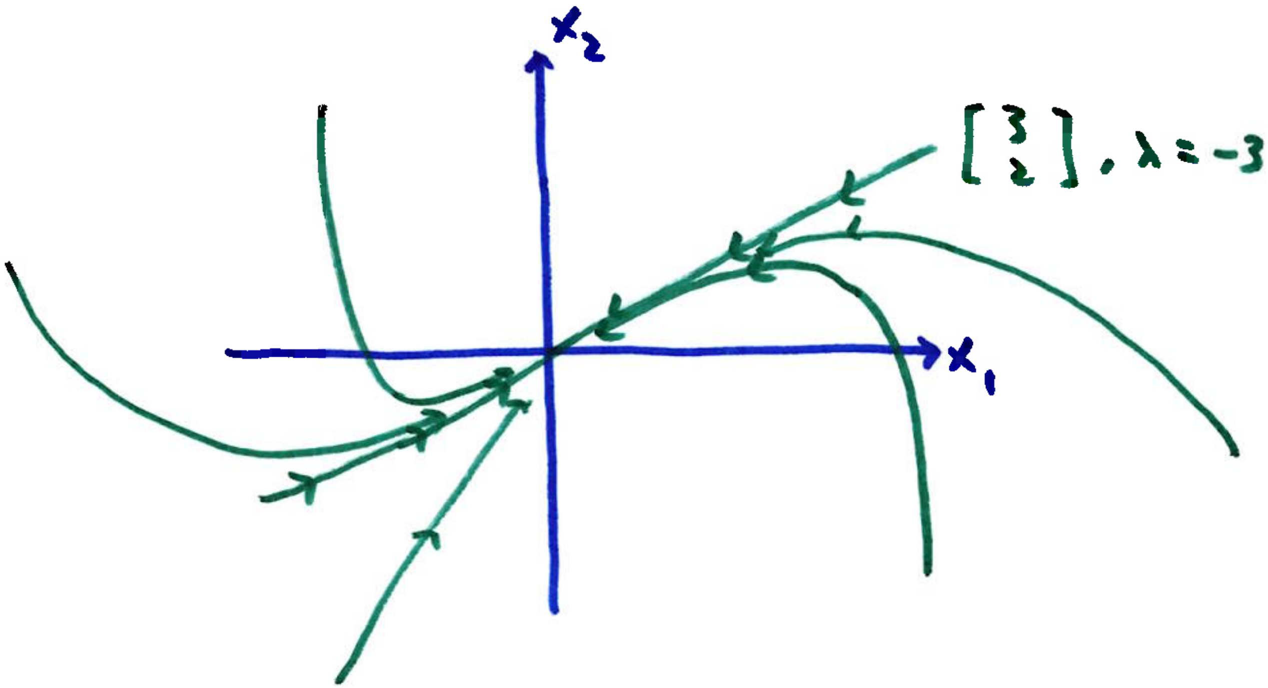
origin is sometimes called a "star node" and is a <u>proper nodal source</u>

no preferred asymptote

$$\vec{x}' = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \vec{x}$$

$\lambda = -3, -3$

$\vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ defect of one



one straight line solution visible

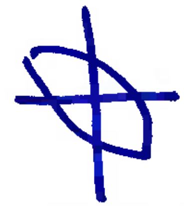
origin is improper nodal sink

Complex λ's

$$\vec{x}' = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \vec{x} \qquad \lambda = i, -i$$

$$\vec{v} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}, \begin{bmatrix} 2-i \\ 1 \end{bmatrix}$$

λ's are purely imaginary, so solution curves are ovals.

Orientation?  or 

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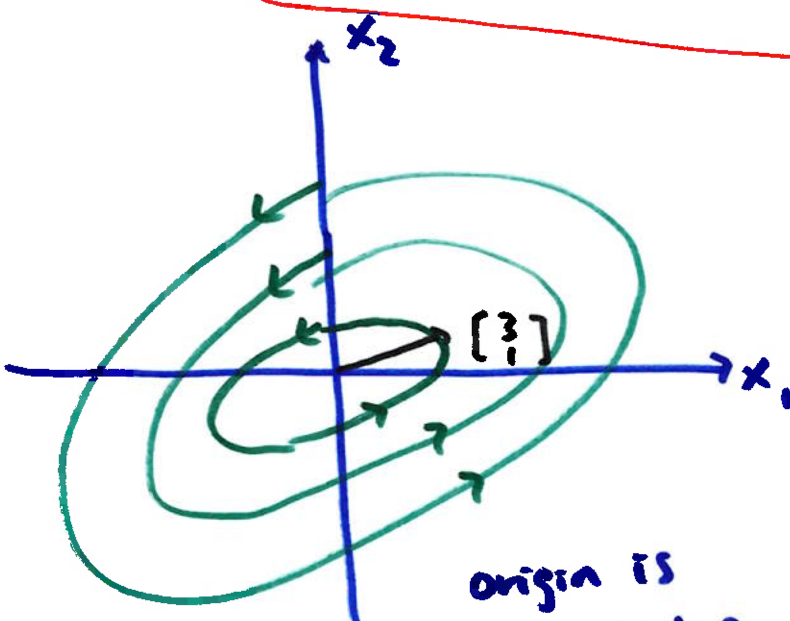
to do that, look at a $\vec{v}$: $\vec{v} = \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Sum of them gives us direction of the major axis



origin is a <u>center</u>

direction: pick $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

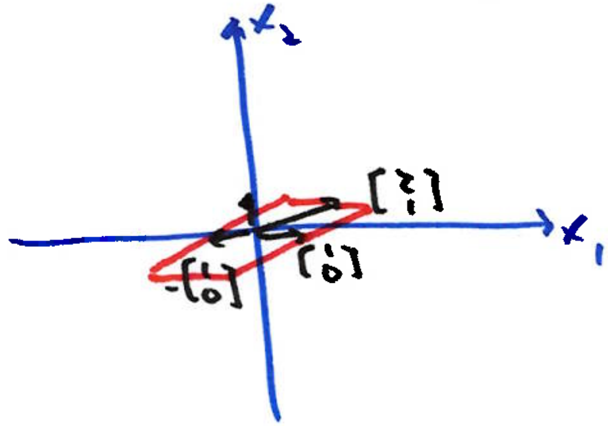
look at $\vec{x}' = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

right, up

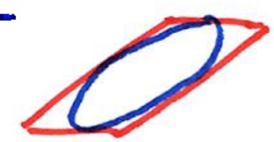
ellipse orientation

eigenvectors $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} - i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

form a parallelogram with $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \pm \begin{bmatrix} 1 \\ 0 \end{bmatrix}$



ellipse is inside the parallelogram



if real part of λ is not zero, then the ovals grow or shrink as they go around

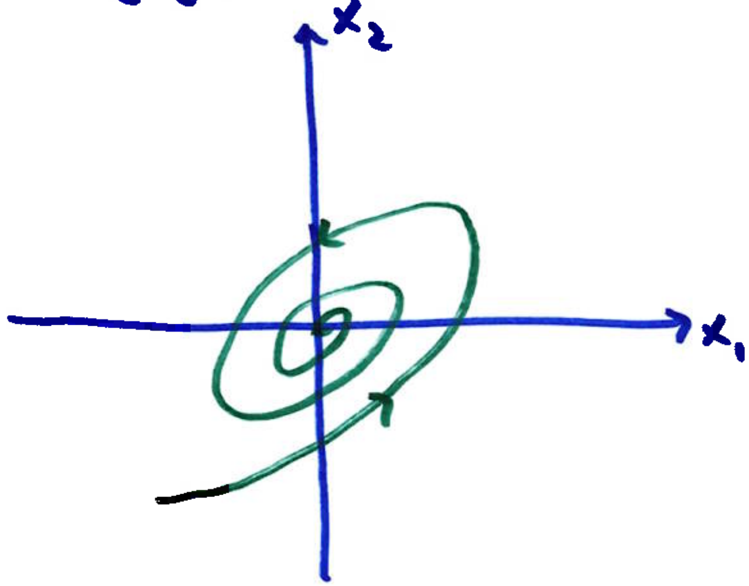
$$\vec{x}' = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} \vec{x}$$

$\lambda = -1+i, \ -1-i$

$\vec{v} = \begin{bmatrix} 2+i \\ 5 \end{bmatrix}, \begin{bmatrix} 2-i \\ 5 \end{bmatrix}$

figure out orientation and direction as before

here, real part of λ is negative, so "oval" shrinks as they go



origin is a <u>spiral sink</u>

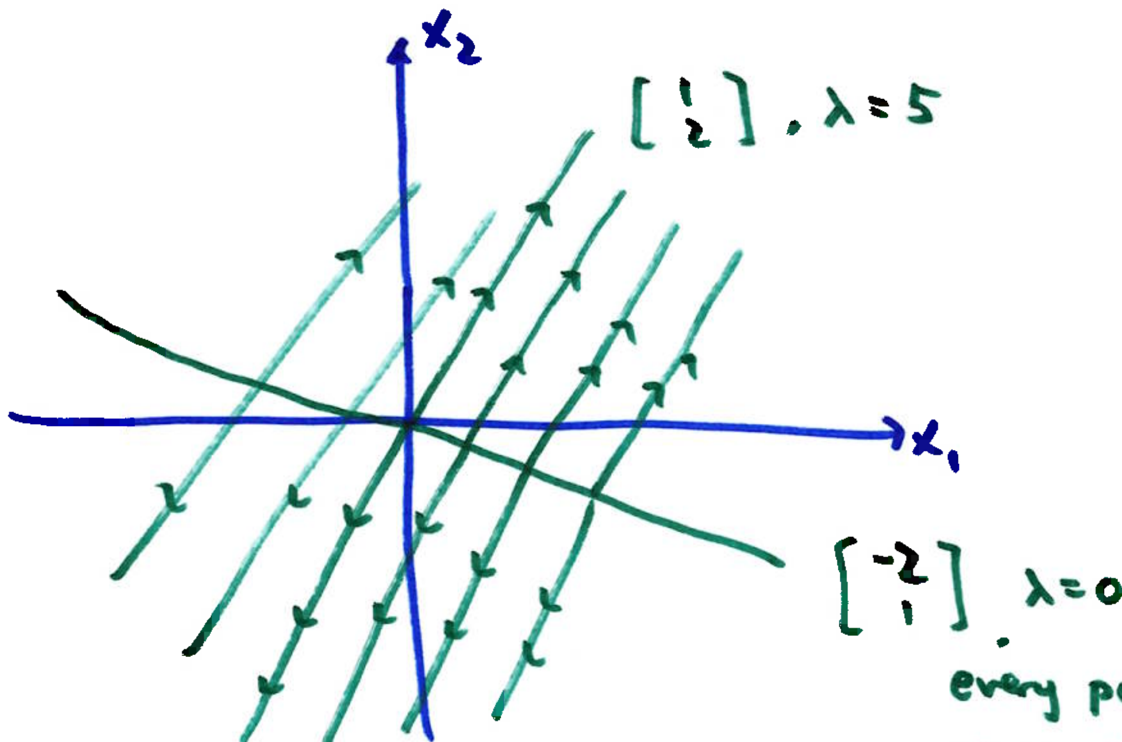
an unusual case: one eigenvalue is zero

$$\vec{x}\,' = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \vec{x}$$

$\lambda = 0,\ 5$

$\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

gen. solution: $\vec{x} = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$



every point on this line
is an equilibrium ($\vec{x}\,' = \vec{0}$)
no directions